0J2 - Mechanics Lecture Notes 1

Mechanics describes the effect of forces on bodies. It is normally divided into two parts:-

Statics: There is no motion, so the forces on the body must balance out – called equilibrium.

Examples are: bridge, car parked on a hill, building, book lying on a table etc. etc.

Dynamics: This deals with systems in motion.

Examples are: cars moving along roads, projectiles, planetary motion, pendulums etc. etc.

We shall consider both statics and dynamics and for each we shall consider

1. Principles: e.g. Newton's Laws.
2. Practice: e.g. the equations describing the motion of a bouncing ball.

Both aspects are important.

Organisation of the course

1. Dynamics I Motion in one dimension (1D) – linear motion. (8 lectures)

2. Dynamics II General motion. (6 lectures)

3. Statics (8 lectures)

Dynamics I. Forces and Motion in 1D

Motion in 1D

We consider particles (with no size) moving along a straight line (i.e. in 1D). We only need one coordinate, x, for the position of the particle.

If the particle starts at x_0 then the change in position, called the *displacement*, is $x - x_0$.

Since we are working in 1D all variables will be scalars not vectors. In general these can also change with time, t. We shall mainly use the following variables

Position	$x(t)$	
Displacement	$s(t)$	$= x(t) - x_0$
Velocity	$v(t)$	$= \dfrac{dx}{dt} = \dfrac{ds}{dt}$
Acceleration	$a(t)$	$= \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt}$

In 1D speed is the positive value of the velocity, and is the same as velocity if the velocity is positive.

Example

A particle moves in 1D with velocity given by

$$v(t) = 5 + 2\cos t \quad \text{ms}^{-1}$$

At $t = 0$ it is at $x = 10$. Find

(i) The maximum magnitude of the acceleration

(ii) The position at time $t = \pi/2$

(iii) The average speed from $t = 0$ to $t = \pi/2$.

Solution

(i) Here $v \;=\; 5 + 2\cos t$ so $a \;=\; \dfrac{dv}{dt} \;=\; -\,2\sin t$

Maximum $|a|$ is when $\sin t \;=\; \pm\,1$. Therefore maximum $|a| \;=\; 2$.

(ii) $\dfrac{dx}{dt} \;=\; v \;=\; 5 + 2\cos t$ so integrating

$$x \;=\; 5t + 2\sin t + k$$

where k is the constant of integration.

$x = 10$ at $t = 0$ so $10 \;=\; 0 + 0 + k \;\;\Rightarrow\;\; k \;=\; 10$. Hence

$$x \;=\; 5t + 2\sin t + 10 \quad \mathrm{ms}^{-1}$$

At $t \;=\; \pi/2$, $\;x \;=\; 5\pi/2 + 2 + 10 \;=\; 19.854$ m.

The change in position from the starting point, the displacement, is $x - 10 \;=\; 9.854$.

The time taken to get to here is $\pi/2$ so the average speed is $\dfrac{9.854}{\pi/2} \;=\; 6.274\ \mathrm{ms}^{-1}$.

Special cases

1. Zero acceleration If the acceleration is zero then $a \;=\; 0$ so $\dfrac{dv}{dt} \;=\; 0$.

Integrating: $v(t) \;=\; u$ where u is the constant of integration. This tells us that the velocity is constant.

Also $\dfrac{dx}{dt} \;=\; u$ so integrating again $x \;=\; k + ut$ where k is the constant of integration.

If $x \;=\; x_0$ at $t = 0$ then $k \;=\; x_0$ so

$$\boxed{x \;=\; x_0 + ut}$$

or, using the displacement, $s = x - x_0$, we have $\boxed{\mathrm{s \;=\; ut}}$. Remember that u is a constant.

This is the equation for uniform motion (i.e. with no acceleration, constant velocity) in 1D

Much more important is

2. Constant acceleration Let the acceleration be constant with value a. Then

$$\frac{dv}{dt} \;=\; a$$

$$v \;=\; \int a\,dt \;=\; k + at$$

where k is the constant of integration.

If $v = u$ at $t = 0$ then $k = u$ so

$$\boxed{\mathrm{v} \;=\; \mathrm{u + at}} \qquad (1)$$

However, $v \;=\; \dfrac{dx}{dt}$ so

$$\begin{aligned} \frac{dx}{dt} &= u + at \\ x &= \int (u + at)\, dt \;=\; c + ut + \frac{at^2}{2} \end{aligned}$$

where c is the constant of integration.

If $x = x_0$ at $t = 0$ then $c = x_0$ and so

$$\boxed{x \;=\; x_0 + ut + \tfrac{1}{2}at^2} \qquad (2)$$

From (1) we have $v - u \;=\; at$ so $\frac{1}{2}at^2 \;=\; \frac{1}{2}(v - u)t$.
Substituting in (2) gives

$$x \;=\; x_0 + ut + \frac{1}{2}(v - u)t$$

$$\boxed{x \;=\; x_0 + \tfrac{1}{2}(u + v)t} \qquad (3)$$

We need one final equation. From (1) $v \;=\; u + at$ so $t \;=\; \dfrac{v - u}{a}$.
Substitute in (3)

$$x - x_0 \;=\; \frac{1}{2}(u + v)\frac{(v - u)}{a}$$

$$\begin{aligned} \therefore \quad 2a(x - x_0) &= (u + v)(v - u) \\ &= v^2 - u^2 \end{aligned}$$

so $$\boxed{v^2 \;=\; u^2 + 2a(x - x_0)} \qquad (4)$$

These four equations describe motion in 1D with constant acceleration.

We usually work with the *displacement* $s \;=\; x - x_0$. The four equations then become

$$\begin{aligned} v &= u + at && (1) \\ s &= ut + \frac{1}{2}at^2 && (2) \\ s &= \frac{1}{2}(u + v)t && (3) \\ v^2 &= u^2 + 2as && (4) \end{aligned}$$

We also need an alternative form of (2) using v instead of u.
From (1) $u = v - at$, and substituting in (2) gives

$$
\begin{aligned}
s &= (v - at)t + \frac{1}{2}at^2 \\
s &= vt - \frac{1}{2}at^2 \qquad \text{(2a)}
\end{aligned}
$$

In these equations the t is the time, s is the displacement, u is the velocity at $t = 0$, v is the velocity at time t and a is the acceleration.

Notes
1. Each equation omits one of the five: t, s, u, v, a.
2. If the acceleration is negative it is usually called the *deceleration.*
3. a and u (and x_0) are constants.
4. It may be necessary to use more than one of these equations.

Example 1 A car travels from rest at a constant acceleration of 6.05 ms^{-2}. How long will it take to travel 1 km and what will be the velocity at the end.

Solution Here $a = 6.05$, $u = 0$ and $s = 1000$ m.
We need t and v. Use (4) to find v:

$$
\begin{aligned}
v^2 &= u^2 + 2as \\
&= 0 + 2 \times 6.05 \times 1000 \\
&= 12100 \\
\therefore \quad v &= \sqrt{12100} = 110 \text{ ms}^{-1}
\end{aligned}
$$

Now use (1) to find t

$$
\begin{aligned}
v &= u + at \\
110 &= 0 + 6.05t \\
\therefore \quad t &= \frac{110}{6.05} = 18.18 \text{ s}
\end{aligned}
$$

Example 2 A ball is rolled along the ground. The initial speed is 16 ms^{-1} and the deceleration is 4 ms^{-2}. How far will it travel?

Solution Here $u = 16$, $v = 0$ and $a = -4$.
We need s so use (4):

$$0^2 = 16^2 + 2 \times (-4) \times s$$

Hence $s = \dfrac{16^2}{8} = 32$ m.

Example 3 A stone is dropped down a (dry!) well and strikes the bottom after 3 s. Neglecting air resistance, at what speed does it land and how deep is the well. Take the acceleration due to gravity $g = 9.81 \text{ ms}^{-2}$.

Solution Measuring downwards we have $u = 0, \quad a = g = 9.81 \quad$ and $t = 3$.
We need v and s.
Using (1) $v = u + at = 30 \text{ ms}^{-1}$.
Using (2) $s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2} \times 9.81 \times 9 = 44.15$ m.

Example 4 A coin is tossed vertically upwards with initial speed $u \text{ ms}^{-1}$. At what time does it reach the highest point and what is the highest point? (Take $g = 9.81 \text{ ms}^{-2}$.)

Solution At the highest point $v = 0$. Measuring upwards we have $a = -9.81$.

Using (1): $v = u + at$ so $0 = u - 9.81t$. Therefore $t = \dfrac{u}{9.81}$.

Now use (3): $s = \frac{1}{2}(u+v)t = \frac{1}{2}(u+0)t = \frac{1}{2}u\dfrac{u}{9.81} = \dfrac{u^2}{19.62}$ m.

We could instead use (4): $s = \dfrac{v^2 - u^2}{2a} = \dfrac{u^2}{19.62}$.

Forces

So far we have not considered the effect of force on moving bodies. The laws which describe these effects are *Newton's Laws*, as follows:

N1 A body will remain at rest *or* in a state of uniform motion in a straight line *unless* it is acted upon by an external force.

(Note: uniform motion means constant velocity.)

N2 When an external force acts on a body of constant mass then it produces an acceleration directly proportional to the force (and in the same direction).

N3 Action and reaction are equal and opposite.
This means that when a body A exerts a force on body B, the B exerts an equal and opposite force on A.

Consequences of Newton's Laws

N1: This law really defines what we mean by a force. The law implies

1. if a body accelerates then the resultant force on it is non-zero,
2. if a body has no acceleration then the resultant force on it is zero,
3. a body can move with a constant velocity if no forces are acting on it.

N2: In 1D the law states that a is proportional to F, so $F = ka$, where k is the constant of proportionality.
Experimentally k is always the same for a given body, and is proportional to the mass of the body, so $F = cma$, where $k = cm$.

In fact, we choose the units of force (called the 'newton') to make $c = 1$, so

$$F = ma$$

1 newton is thus the force needed to accelerate a mass of 1 kg at $1\,\mathrm{ms}^{-2}$.

Note that N1 tells us that if $F = 0$ then $v =$ constant, i.e. $v = k$ (k constant).
Therefore $dv/dt = 0$. But $a = dv/dt$ so $a = 0$ as expected from N2.
This shows that N1 and N2 are consistent.

N3 No further comments needed.

Force due to Gravity

Galileo showed that all bodies falling freely, vertically, have the same acceleration g, independent of mass m. $g = 9.81\mathrm{ms}^{-2}$. (We have already used this above.)

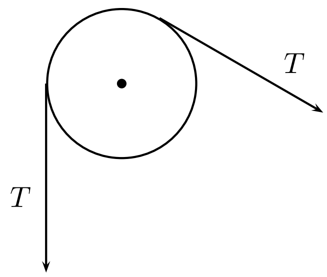
By N2 this means that the force of gravity must be

$$F = mg$$

for a body of mass m.

Pulleys

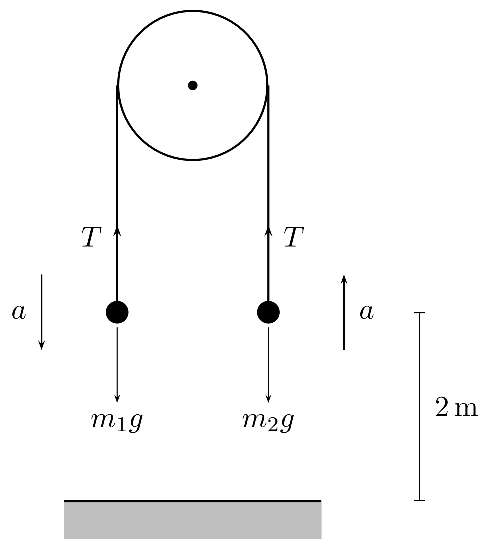
A pulley is a smooth (i.e. frictionless) disc, about which a string can pass. The magnitude of the tension (i.e. force) T in the string is the same on both sides of the pulley. Clearly the direction is different.



Alternatively, the disc may be free to rotate without friction about its centre. Again there is no change in the magnitude of T, only direction.

Example 1 Particles of masses m_1 and m_2 kg, with $m_1 > m_2$ are attached to either end of a light inelastic string which passes over a pulley. The two particles are initially at rest, at a height 2 m above a table. Find the acceleration of m_1, and the velocity with which it hits the table.

Let the tension in the string be T.



Apply N2 to m_1 particle (measure distances *downwards*)

$$m_1 g - T = m_1 a \tag{1}$$

Apply N2 to m_2 particle (*upwards*)

$$T - m_2 g = m_2 a \tag{2}$$

Add (1) and (2): $(m_1 - m_2)g = (m_1 + m_2)a$, so

$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$

Now $u = 0$, $s = 2$, $a =$ as above, and we want v.

$v^2 = u^2 + 2as$ gives $v^2 = 0 + 2\,\dfrac{m_1 - m_2}{m_1 + m_2} g \times 2$ so

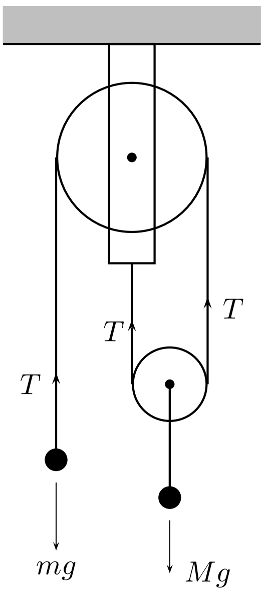
$$v = 2\sqrt{\frac{m_1 - m_2}{m_1 + m_2}\, g}$$

N.B. This system is not in equilibrium which means that the total force on each particle is not zero and the acceleration is also not zero. Thus $T \neq m_1 g$ and $T \neq m_2 g$. We shall discuss systems in equilibrium in the Statics section of the course later.

Example 2a

Consider two pulleys connected as shown. What mass m is needed is needed to balance the mass M?

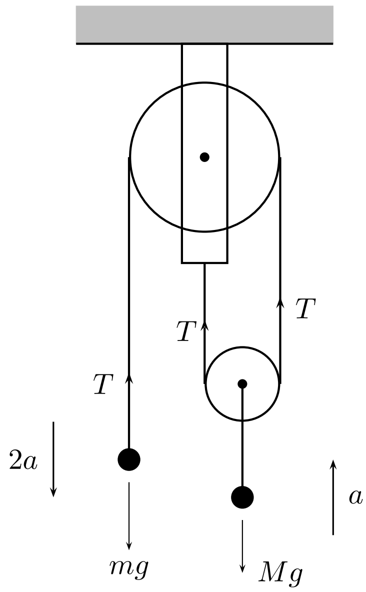
The centre of the large pulley is fixed and so is the end of the string which is below the large pulley.



If the two masses are balanced then there is no acceleration. Since $F = ma$ there is no total force F on either mass.

For mass m this means that $T - mg = 0$ so $T = mg$.
For mass M this means that $2T - Mg = 0$ so $2mg - Mg = 0$ and so $m = \frac{1}{2}M$.

Example 2b



Find the acceleration of mass M in the same configuration as Ex 2a but with $m = \dfrac{3M}{4}$.

Here, if mass M moves a distance x upwards, then mass m moves a distance $2x$ downwards.

Thus if the acceleration of M is $a = \ddot{x}$ upwards then the acceleration of m is $2a = 2\ddot{x}$ downwards.

Using N2 for mass m: $\quad mg - T = m(2a) \quad (1) \quad$ (downwards).
Using N2 for mass M: $\quad 2T - Mg = Ma \quad (2) \quad$ (upwards).

$2 \times (1) + (2)$ gives: $\quad 2mg - Mg = 2m(2a) + Ma = 4ma + Ma$.

$$\therefore \quad a = \frac{2mg - Mg}{4m + M}$$

But $m = \dfrac{3M}{4}$ so

$$a = \frac{\frac{3}{2}Mg - Mg}{3M + M} = \frac{\frac{1}{2}Mg}{4M} = \frac{1}{8}g\,.$$